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EQUATIONS OF MOTION OF A MISSILE AND A SATELLITE FOR AN OBLATE-SPHEROIDAL ROTATING EARTH

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EQUATIONS OF MOTION OF A MISSILE AND A SATELLITE FOR AN OBLATE-SPHEROIDAL ROTATING EARTH

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Pasadena, California
April 12, 1957

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ABSTRACT

The equations of motion of a rocket are derived by applying the fundamental definition of the derivative to Newton's second law of motion. Three independent cases are considered: motion of the missile center of mass, rotation of the missile about a transverse axis through the center of mass, and rotation of the missile about the longitudinal axis. The second case describes the motion in pitch (or yaw), and the third case describes the rotation (spin or roll) of the missile due to a ring of small jets placed around its circumference. The equations of motion of the center of mass are then modified to describe the motion of a satellite moving around the earth in a nearly circular orbit. Finally, a method is developed for computing the approximate impact point of the missile by algebraic means.

I. INTRODUCTION

The purpose of this Memorandum is to derive the equations of motion of a missile, for an oblate-spheroidal rotating earth, from elementary concepts and to express these equations in a convenient and na ural coordinate system. Spherical polar coordinates are used to locate the position of the missile center of mass relative to the earth. It is assumed that the earth is shaped like a perfect sphere, but that its gravitational field is that corresponding to an oblate spheroid.

The motion of a satellite is determined directly from the equations of motion of the missile, the former being considered a special case of the latter.

II. MOTION OF CENTER OF MASS

Consider the origin in Fig. 1 to coincide with the center of the earth. Let the set of axes $\Sigma\,(x, y, z)$[a] be fixed in inertial space, and let the set $\Sigma'(x', y', z')$ be rigidly attached to the earth. Consider that Σ' rotates at a constant angular velocity $\vec{\Omega}$ with respect to Σ.

In Fig. 1, m_1 represents the mass of a quantity of exhaust expelled from the missile. The center of mass (C. M.) of m_1 and the missile is located at P_1, whereas the C. M. of the missile is located at P_2. The position vectors $\vec{s}$ and $\vec{\rho}$ are drawn from P_1 to m_1 and P_2, respectively. It is assumed that $\vec{s} \times \vec{\rho} = 0$. The position vectors $\vec{r}_1$, $\vec{R}$, and $\vec{r}$ are drawn from the origin to m_1, P_1, and P_2, respectively. Let $t = t_0$ be the instant of time just before, and $t = t_0 + h$ the instant of time just after, m_1 is expelled. Here, $h > 0$ is an infinitesimal time. Thus,

$$\vec{\rho}(t_0) = 0 \tag{1a}$$

$$\vec{r}(t_0) = \vec{R}(t_0) \tag{1b}$$

$$\vec{r}(t_0 + h) = \vec{R}(t_0 + h) + \vec{\rho}(t_0 + h) \tag{1c}$$

$$\vec{r}_1(t_0 + h) = \vec{R}(t_0 + h) + \vec{s}(t_0 + h) \tag{1d}$$

$$m_1 = m(t_0) - m(t_0 + h) \tag{1e}$$

where $m(t_0)$ and $m(t_0 + h)$ refer to the mass of the missile.

Let $\vec{p}$ represent the total linear momentum as measured in Σ. Then

$$\vec{p}(t_0) = m(t_0)\,\frac{d}{dt}\vec{r}(t_0)$$

$$\vec{p}(t_0 + h) = m_1\,\frac{d}{dt}\vec{r}_1(t_0 + h) + m(t_0 + h)\,\frac{d}{dt}\vec{r}(t_0 + h)$$

From Eqs. (1),

$$\vec{r}_1(t_0 + h) = \vec{r}(t_0 + h) + \vec{s}(t_0 + h) - \vec{\rho}(t_0 + h)$$

---

[a] The nomenclature used throughout this Memorandum is defined in Table 1.

Substituting this and Eq. (1e) into the expressions for $\vec{p}(t_0)$ and $\vec{p}(t_0 + h)$ yields

$$\vec{p}(t_0 + h) - \vec{p}(t_0) = m(t_0)\left[\frac{d}{dt}\vec{r}(t_0 + h) - \frac{d}{dt}\vec{r}(t_0)\right]$$

$$+ \left[m(t_0) - m(t_0 + h)\right]\frac{d}{dt}\left[\vec{s}(t_0 + h) - \vec{\rho}(t_0 + h)\right]$$

Now,

$$\vec{F} = \textit{external}\text{ forces acting on the system}$$

$$= \lim_{h \to 0}\frac{\vec{p}(t_0 + h) - \vec{p}(t_0)}{h}$$

$$= m(t_0)\lim_{h \to 0}\frac{\frac{d}{dt}\vec{r}(t_0 + h) - \frac{d}{dt}\vec{r}(t_0)}{h}$$

$$- \lim_{h \to 0}\frac{m(t_0 + h) - m(t_0)}{h}\cdot\frac{d}{dt}\left[\vec{s}(t_0 + h) - \vec{\rho}(t_0 + h)\right]$$

$$\vec{F} = m(t_0)\frac{d^2}{dt^2}\vec{r}(t_0) - \dot{m}(t_0)\frac{d}{dt}\left[\vec{s}(t_0) - \vec{\rho}(t_0)\right] \tag{2}$$

In Eq. (2), d/dt signifies rate of change in Σ. The quantities $\vec{F}, \vec{r}, \vec{s}$, and $\vec{\rho}$ can be expressed in either set of coordinates. Since it is most likely that $\vec{F}$ and $\vec{r}$ will be measured in Σ', it will prove more expedient to express Eq. (2) in terms of rate of change in Σ': i.e., in terms of d'/dt. Now,

$$\frac{d\vec{r}}{dt} = \frac{d'\vec{r}}{dt} + \vec{\Omega} \times \vec{r}$$

and similarly for $\vec{s}$ and $\vec{\rho}$. Therefore, Eq. (2) can be written

$$\vec{F} = m(t_0)\left[\frac{d'^2}{dt^2}\vec{r}(t_0) + 2\vec{\Omega}\times\frac{d'}{dt}\vec{r}(t_0) + \vec{\Omega}\times\vec{\Omega}\times\vec{r}(t_0)\right]$$

$$- \dot{m}(t_0)\left\{\frac{d'}{dt}\left[\vec{s}(t_0) - \vec{\rho}(t_0)\right] + \vec{\Omega}\times\vec{s}(t_0)\right\} \qquad (3)$$

since $\vec{\Omega}$ = a constant and $\vec{\rho}(t_0) = 0$. Let

$$\frac{d'}{dt}\left[\vec{s}(t_0) - \vec{\rho}(t_0)\right] = \vec{c}$$

where $\vec{c}$ is the velocity of m_1 with respect to the missile (exhaust velocity). Now, $|\vec{\Omega}\times\vec{s}(t_0)| \leq |\vec{\Omega}|\ |\vec{s}(t_0)|$. For a typical missile, $|\vec{s}(t_0)| \simeq 20$ ft. Since $|\vec{\Omega}| = 7.27\times 10^{-5}$ radians/sec, then $|\vec{\Omega}\times\vec{s}(t_0)| \leq 0.00145$ ft/sec, which is negligible compared with $|\vec{c}| \simeq 2000$ ft/sec. Hence, the last term in Eq. (3) can be omitted. Since no restrictions are placed on the time t_0 except that it occur before m_1 is expelled, and since m_1 is being continually expelled (during burning), then

$$\vec{F} = m(t)\left[\frac{d'}{dt}\vec{v}(t) + 2\vec{\Omega}\times\vec{v}(t) + \vec{\Omega}\times\vec{\Omega}\times\vec{r}(t)\right] - \dot{m}(t)\,\vec{c} \qquad (4)$$

where $\vec{v}(t) = (d'/dt)\vec{r}(t)$. Here, $\vec{r}(t)$ and $\vec{F}$ are measured in Σ'. The quantity $\vec{F}$ represents the sum of all external forces acting on the missile (excluding vacuum thrust), and the quantity $+\dot{m}(t)\,\vec{c}$ equals the vacuum thrust of the rocket, where $\dot{m} < 0$. The quantities $-2m\,\vec{\Omega}\times\vec{v}$ and $-m\,\vec{\Omega}\times\vec{\Omega}\times\vec{r}$ are the Coriolis and centrifugal forces, respectively.

III. ROTATION ABOUT A TRANSVERSE AXIS

Let $\vec{L}$ be the total angular momentum about the origin (see Fig. 1). Then

$$\vec{L}(t_0) = \vec{R}(t_0) \times m(t_0) \frac{d}{dt} \vec{R}(t_0) + I(t_0)\vec{\gamma}(t_0)$$

$$\vec{L}(t_0 + h) = \vec{R}(t_0 + h) \times m(t_0) \frac{d}{dt} \vec{R}(t_0 + h) + \vec{s}(t_0 + h) \times m_1 \frac{d}{dt} \vec{s}(t_0 + h)$$

$$+ \vec{\rho}(t_0 + h) \times m(t_0 + h) \frac{d}{dt} \vec{\rho}(t_0 + h) + I(t_0 + h)\vec{\gamma}(t_0 + h)$$

where $\vec{\gamma}$ = angular velocity about P_2, and I = moment of inertia about a transverse axis through the same point. Let the origin coincide with P_2, so that $\vec{R}(t_0 + h) = -\vec{\rho}(t_0 + h)$, and $\vec{R}(t_0) = 0$. Then

$$\vec{L}(t_0 + h) - \vec{L}(t_0) = \left[m(t_0 + h) + m(t_0)\right] \vec{\rho}(t_0 + h) \times \frac{d}{dt} \vec{\rho}(t_0 + h)$$

$$- \left[m(t_0 + h) - m(t_0)\right] \vec{s}(t_0 + h) \times \frac{d}{dt} \vec{s}(t_0 + h)$$

$$+ I(t_0 + h)\vec{\gamma}(t_0 + h) - I(t_0)\vec{\gamma}(t_0)$$

Now

$$\vec{M} = \text{torques taken about } P_2$$

$$= \lim_{h \to 0} \frac{\vec{L}(t_0 + h) - \vec{L}(t_0)}{h}$$

$$M = \lim_{h \to 0} \frac{\vec{\rho}(t_0 + h) - \vec{\rho}(t_0)}{h} \times \frac{d}{dt}\vec{\rho}(t_0 + h)\left[m(t_0 + h) + m(t_0)\right]$$

$$- \lim_{h \to 0} \frac{m(t_0 + h) - m(t_0)}{h} \vec{s}(t_0 + h) \times \frac{d}{dt}\vec{s}(t_0 + h)$$

$$+ \lim_{h \to 0} \frac{I(t_0 + h)\vec{\gamma}(t_0 + h) - I(t_0)\vec{\gamma}(t_0)}{h}$$

since $\vec{\rho}(t_0) = 0$.

$$\vec{M} = \frac{d}{dt}\vec{\rho}(t_0) \times \frac{d}{dt}\vec{\rho}(t_0)\left[2m(t_0)\right] - \dot{m}(t_0)\vec{s}(t_0) \times \frac{d}{dt}\vec{s}(t_0) + \frac{d}{dt} I(t_0)\vec{\gamma}(t_0)$$

The first term on the right is obviously zero. Now, $\vec{s}(t_0) = \vec{s}_1\, s(t_0)$, where $\vec{s}_1$ = unit vector. Then

$$\frac{d}{dt}\vec{s}(t_0) = \vec{s}_1 \frac{d}{dt} s(t_0) + s(t_0)\,(\vec{\gamma} \times \vec{s}_1)$$

where $\vec{\gamma} = \gamma\vec{\phi}_1$. This is illustrated in Fig. 2, where $\vec{\phi}_1$ is a unit vector pointing into the paper, and $\vec{\theta}_1$ is a unit vector in the direction of changing $\vec{s}_1$. Then $\vec{\gamma} \times \vec{s}_1 = \gamma(\vec{\phi}_1 \times \vec{s}_1) = \gamma\vec{\theta}_1$, so that

$$\vec{s}(t_0) \times \frac{d}{dt} s(t_0) = s(t_0)\,\vec{s}_1 \times \left[\dot{s}(t_0)\vec{s}_1 + \gamma s(t_0)\vec{\theta}_1\right]$$

$$= \gamma s^2(t_0)\,\vec{s}_1 \times \vec{\theta}_1$$

$$= s^2(t_0)\,\gamma\vec{\phi}_1$$

$$= s_0^2\,\vec{\gamma}$$

Therefore,

$$\vec{M} = \frac{d}{dt}\left[I(t)\,\vec{\gamma}(t)\right] - \dot{m}(t)\,s_0^2\,\vec{\gamma} \tag{5}$$

Assigning a value to s_0 is difficult, since s_0 = distance from P_2 (at time $t = t_0$) to m_1, and since the exact location of m_1 at $t = t_0$ is not known. Let s_0 be the distance from P_2 (at $t = t_0$) to the base of the missile.

IV. ROTATION ABOUT THE LONGITUDINAL AXIS

The rapid rotation of a missile about its longitudinal axis can produce the same stability in flight as that achieved by a missile employing fin deflection. This rotation is generated by two or more jets situated around the circumference of the missile at the rear. In the present discussion, it is assumed that, if $\vec{f}_i$ = thrust of the ith jet, then $\sum_{i=1}^{N} \vec{f}_i = 0$, where N = total number of jets.

Figure 3 shows the rear of the missile. The longitudinal axis passes through the center of the circle (representing the missile) and is perpendicular to the paper. In Fig. 3(a), the position vector $\vec{r}_i$, drawn perpendicular to the longitudinal axis, points to the ith jet at time $t = t_0$. Figure 3(b) shows the location of this jet at time $t = t_0 + h$. The quantity m_i is the mass of a quantity of exhaust expelled from the ith jet, and $\vec{s}_i$, $\vec{R}_i$ are position vectors locating m_i, as shown. The plane determined by $\vec{r}_i$, $\vec{s}_i$, $\vec{R}_i$ is perpendicular to the axis. From Fig. 3,

$$\left.\begin{aligned} \vec{s}_i(t_0) &= 0 \\ \vec{R}_i(t_0) &= \vec{r}_i(t_0) \\ \vec{R}_i(t_0 + h) &= \vec{r}_i(t_0 + h) + \vec{s}_i(t_0 + h) \end{aligned}\right\} \tag{6}$$

Let $\vec{L}$ represent the total angular momentum. Then

$$\vec{L}(t_0) = I(t_0)\,\vec{\omega}(t_0)$$

$$\vec{L}(t_0 + h) = I(t_0 + h)\,\vec{\omega}(t_0 + h) + \sum_{i=1}^{N} \vec{R}_i(t_0 + h) \times m_i \frac{d}{dt} \vec{R}_i(t_0 + h)$$

where $\vec{\omega}$ = angular velocity about the longitudinal axis, and I = moment of inertia about this axis. The second term in $\vec{L}(t_0 + h)$ is the angular momentum of all the m_i about the origin. If it is assumed that $m_1 = m_2 = \cdots = m_N$, then

$$\sum_{i=1}^{N} \vec{R}_i(t_0 + h) \times m_i \frac{d}{dt} \vec{R}_i(t_0 + h) = m_N \sum_{i=1}^{N} \left[\vec{r}_i(t_0 + h) \times \frac{d}{dt} \vec{r}_i(t_0 + h)\right.$$

$$+ \vec{r}_i(t_0 + h) \times \frac{d}{dt} \vec{s}_i(t_0 + h) + \vec{s}_i(t_0 + h)$$

$$\left.\times \frac{d}{dt} \vec{r}_i(t_0 + h) + \vec{s}_i(t_0 + h) \times \frac{d}{dt} \vec{s}_i(t_0 + h)\right]$$

Now, $m(t_0) = m(t_0 + h) + N m_N$, where m refers to the mass of the missile. But

$$\frac{d}{dt} m(t_0) = \lim_{h \to 0} \frac{m(t_0 + h) - m(t_0)}{h}$$

so that

$$\frac{d}{dt} m(t_0) = \dot{m}(t_0) = -N \lim_{h \to 0} \frac{m_N}{h}$$

Let $\vec{T}$ = torques taken about the longitudinal axis. Then, since $s_i(t_0) = 0$,

$$\mathrm{T} = \lim_{h \to 0} \frac{\vec{L}(t_0 + h) - \vec{L}(t_0)}{h}$$

$$= \lim_{h \to 0} \frac{I(t_0 + h)\,\vec{\omega}(t_0 + h) - I(t_0)\,\vec{\omega}(t_0)}{h}$$

$$+ \lim_{h \to 0} \frac{m_N}{h} \sum_{i=1}^{N} \left[\vec{r}_i \times \frac{d\vec{r}_i}{dt} + \vec{r}_i \times \frac{d\vec{s}_i}{dt} + \vec{s}_i \times \frac{d\vec{r}_i}{dt} + \vec{s}_i \times \frac{d\vec{s}_i}{dt} \right]$$

$$= \frac{d}{dt}\left[I(t_0)\,\vec{\omega}(t_0) \right] - \frac{\dot{m}(t_0)}{N} \sum_{i=1}^{N} \vec{r}_i(t_0) \times \left[\frac{d}{dt}\vec{r}_i(t_0) + \frac{d}{dt}\vec{s}_i(t_0) \right]$$

Hence,

$$\vec{\mathrm{T}} = \frac{d}{dt}\left[I(t)\,\vec{\omega}(t) \right] - \frac{\dot{m}(t)}{N} \sum_{i=1}^{N} \vec{r}_i(t) \times \left[\frac{d}{dt}\vec{r}_i(t) + \frac{d}{dt}\vec{s}_i(t) \right] \tag{7}$$

But $\vec{r}_i = \vec{r}_{0_i} r$, where $\vec{r}_{0_i}$ = unit vector and r is constant. Also, $\vec{\omega} = \vec{\omega}_0\, \omega(t)$, where $\vec{\omega}_0$ lies along the longitudinal axis (see Fig. 4). Hence, $(d/dt)\,\vec{r}_i = \vec{\omega} \times \vec{r}_i = \omega r\, \vec{\theta}_{0_i}$, where $\vec{\theta}_{0_i}$ is a unit vector in the direction of changing $\vec{r}_i$. The velocity at which m_i leaves the jet (exhaust velocity) is equal to $d\vec{s}_i/dt$. From Fig. 4,

$$\frac{d\vec{s}_i}{dt} = \left| \frac{d\vec{s}_i}{dt} \right| \left(-\vec{\theta}_{0_i} \right)$$

$$= -v_e\, \vec{\theta}_{0_i}$$

Therefore,

$$\sum_{i=1}^{N} \vec{r}_i(t) \times \left[\frac{d}{dt}\vec{r}_i(t) + \frac{d}{dt}\vec{s}_i(t)\right] = Nr(\omega r\vec{\omega}_0 - v_e\vec{\omega}_0)$$

so that

$$\vec{T} = \frac{d}{dt}\left[I(t)\,\omega(t)\right]\vec{\omega}_0 - \dot{m}(t)\,r(\omega r - v_e)\,\vec{\omega}_0 \tag{8}$$

Here, $\omega(t)$, rather than $\int\omega(t)\,dt$, is of chief interest. The torques $\vec{T}$ are highly dependent on the particular design of the jets.

V. EQUATIONS OF MOTION

A. Missile

In Fig. 5, the coordinate axes x, y, z are fixed in the earth, and the origin coincides with the center (of mass) of the earth. The quantities $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors pointing in the directions of increasing x, y, z, respectively. The angular velocity vector $\vec{\Omega}$ of the earth lies along the positive y axis, as shown. The position coordinates of the center of mass (C. M.) of the missile are r, ϕ, θ, defined by Eqs. (9) and Fig. 5, as follows:

$$\left.\begin{aligned} x &= r\cos\theta\cos\phi \\ y &= r\sin\phi \\ z &= r\sin\theta\cos\phi \end{aligned}\quad \begin{pmatrix} \frac{-\pi}{2} \le \phi \le \frac{\pi}{2} \\ -\pi \le \theta \le \pi \end{pmatrix}\right\} \tag{9}$$

The velocity coordinates of the C. M. are v, Θ, σ, defined in Fig. 6. Here, $\vec{r}_0$, $\vec{\phi}_0$, $\vec{\theta}_0$ are unit vectors pointing in the directions of increasing r, ϕ, θ, respectively. The angle between the velocity vector $\vec{v}$ and $\vec{r}_0$ is $(\pi/2) - \Theta$; σ is the angle between $\vec{\phi}_0$ and the projection of $\vec{v}$ in the plane normal to $\vec{r}_0$. From Fig. 6,

$$\vec{r} = \vec{r}_0 \, r \tag{10a}$$

$$\vec{v} = \vec{r}_0 \, v \sin \Theta + \vec{\phi}_0 \, v \cos \Theta \cos \sigma + \vec{\theta}_0 \, v \cos \Theta \sin \sigma \tag{10b}$$

Now,

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{r}_0 \, \dot{r} + r \frac{d\vec{r}_0}{dt}$$

but

$$\left.\begin{aligned} \vec{r}_0 &= \vec{i} \cos \theta \cos \phi + \vec{j} \sin \phi + \vec{k} \sin \theta \cos \phi \\ \vec{\phi}_0 &= \vec{i}(-\cos \theta \sin \phi) + \vec{j} \cos \phi + \vec{k}(-\sin \theta \sin \phi) \\ \vec{\theta}_0 &= \vec{i}(-\sin \theta) + \vec{k} \cos \theta \end{aligned}\right\} \tag{11}$$

Hence,

$$\frac{d\vec{r}_0}{dt} = \vec{\phi}_0 \, \dot{\phi} + \vec{\theta}_0 \cos \phi \, \dot{\theta}$$

so that

$$\vec{v} = \vec{r}_0 \, \dot{r} + \vec{\phi}_0 \, r \dot{\phi} + \vec{\theta}_0 \, r \cos \phi \, \dot{\theta} \tag{12}$$

Equating Eqs. (10b) and (12) yields

$$\dot{r} = v \sin \Theta \tag{13a}$$

$$\dot{\phi} = \frac{v}{r} \cos \Theta \cos \sigma \tag{13b}$$

$$\dot{\theta} = \frac{v}{r} \cos \Theta \frac{\sin \sigma}{\cos \phi} \tag{13c}$$

From Eq. (10b) and Eqs. (11),

$$\begin{aligned}\frac{d\vec{v}}{dt} = {} & \vec{r}_0 \left(v \cos \Theta \dot{\Theta} + \dot{v} \sin \Theta - \frac{v^2}{r} \cos^2 \Theta\right) + \vec{\phi}_0 \Big(-v \cos \Theta \sin \sigma \dot{\sigma} - v \cos \sigma \sin \Theta \dot{\Theta} \\ & + \dot{v} \cos \Theta \cos \sigma + \frac{v^2}{r} \sin \Theta \cos \Theta \cos \sigma + \frac{v^2}{r} \cos^2 \Theta \sin^2 \sigma \tan \phi\Big) \\ & + \vec{\theta}_0 \Big(v \cos \Theta \cos \sigma \dot{\sigma} - v \sin \sigma \sin \Theta \dot{\Theta} + \dot{v} \cos \Theta \sin \sigma + \frac{v^2}{r} \sin \Theta \cos \Theta \sin \sigma \\ & - \frac{v^2}{r} \cos^2 \Theta \cos \sigma \sin \sigma \tan \phi\Big)\end{aligned} \tag{14}$$

The angular velocity of the earth is given by

$$\vec{\Omega} = \vec{j}\Omega = \vec{r}_0 \Omega \sin \phi + \vec{\phi}_0 \Omega \cos \phi$$

Hence,

$$\begin{aligned}\vec{\Omega} \times \vec{v} = {} & \vec{r}_0 \Omega v \cos \Theta \cos \phi \sin \sigma + \vec{\phi}_0 (-\Omega v \cos \Theta \sin \phi \sin \sigma) \\ & + \vec{\theta}_0 \Omega v (\cos \Theta \sin \phi \cos \sigma - \sin \Theta \cos \phi)\end{aligned} \tag{15}$$

$$\vec{\Omega} \times \vec{\Omega} \times \vec{r} = \vec{r}_0 \left(-\Omega^2 r \cos^2 \phi\right) + \vec{\phi}_0 \, \Omega^2 r \sin \phi \cos \phi \tag{16}$$

The external forces $\vec{F}$ acting on the C. M. will, in general, consist of the following:

1. Aerodynamic forces

   $\vec{B}$ = base drag effect
   $\vec{D}$ = drag
   $\vec{L}$ = lift (excluding rudder)
   $\vec{N}$ = lift of rudder

2. Nonaerodynamic forces

   $\vec{J}$ = jet vane and/or exhaust obstruction effect
   $m\vec{g}$ = gravity

It is assumed that the vacuum thrust $\vec{f}_0$ and all the forces listed above, excluding gravity, lie in the plane determined by $\vec{v}$ and $\vec{r}_0$, as shown in Fig. 7. Here, $\vec{f}_0 \times \vec{B} = \vec{f}_0 \times \vec{J} = \vec{v} \times \vec{D} = \vec{L} \cdot \vec{D} = 0$, and $\vec{N}$ is normal to the longitudinal axis of the missile. From Figs. 6 and 7,

$$\left.\begin{aligned}
\vec{f}_0 &= \vec{r}_0 f_0 \sin(\Theta + \alpha + \eta) + \vec{\phi}_0 f_0 \cos(\Theta + \alpha + \eta) \cos\sigma + \vec{\theta}_0 f_0 \cos(\Theta + \alpha + \eta) \sin\sigma \\
\vec{B} &= -\frac{B}{f_0} \vec{f}_0 \\
\vec{J} &= -\frac{J}{f_0} \vec{f}_0 \\
\vec{D} &= -\frac{D}{v} \vec{v} \\
\vec{L} &= \vec{r}_0 L \cos\Theta - \vec{\phi}_0 L \sin\Theta \cos\sigma - \vec{\theta}_0 L \sin\Theta \sin\sigma \\
\vec{N} &= -\vec{r}_0 N \cos(\Theta + \alpha) + \vec{\phi}_0 N \sin(\Theta + \alpha) \cos\sigma + \vec{\theta}_0 N \sin(\Theta + \alpha) \sin\sigma
\end{aligned}\right\} \tag{17}$$

The potential function for an oblate-spheroid earth[a] is

$$V(r,\phi) = \frac{GMm}{R}\left[\frac{R}{r} + K_1 \frac{R^3}{r^3}\left(\frac{1}{3} - \sin^2\phi\right) + \frac{K_2}{35}\frac{R^5}{r^5}\left(35\sin^4\phi - 30\sin^2\phi + 3\right)\right] \tag{18}$$

where: G = universal gravition constant = 6.664×10^{-8} dyne cm^2/gm^2; M = mass of the earth = 5.975×10^{27} gm; R = equatorial radius of the earth = 6.3783×10^{8} cm; $K_1 = 1.637 \times 10^{-3}$; and $K_2 = 1.07 \times 10^{-5}$. The force of gravity is then given by

$$m\vec{g} = -\nabla V(r,\phi) = -\left[\vec{r}_0 \frac{\partial}{\partial r} + \vec{\phi}_0 \frac{1}{r}\frac{\partial}{\partial \phi} + \vec{\theta}_0 \frac{1}{r\cos\phi}\frac{\partial}{\partial\theta}\right] V(r,\phi)$$

or

$$\vec{g} = -\vec{r}_0 g_1(r,\phi) - \vec{\phi}_0 g_2(r,\phi) \tag{19}$$

where

$$g_1(r,\phi) = \frac{GM}{R}\left[\frac{R}{r^2} + 3K_1 \frac{R^3}{r^4}\left(\frac{1}{3} - \sin^2\phi\right) + \frac{K_2}{7}\frac{R^5}{r^6}\left(35\sin^4\phi - 30\sin^2\phi + 3\right)\right] \tag{20}$$

$$g_2(r,\phi) = \frac{GM}{R}\left[K_1 - 2K_2\frac{R^2}{r^2}\left(\sin^2\phi - \frac{3}{7}\right)\right]\left(2\frac{R^3}{r^4}\sin\phi\cos\phi\right) \tag{21}$$

The quantities K_1 and K_2 are a measure of the earth's oblateness. When $K_1 = K_2 = 0$, $\vec{g} = -\vec{r}_0\,(GM/r^2)$.

Equation (4) can be written

$$\vec{B} + \vec{D} + \vec{L} + \vec{N} + \vec{T} + m\vec{g} + \vec{f}_0 = m\left(\frac{d\vec{v}}{dt} + 2\vec{\Omega}\times\vec{v} + \vec{\Omega}\times\vec{\Omega}\times\vec{r}\right) \tag{22}$$

[a] Jeffreys, H., *The Earth: Its Origin, History and Physical Constitution*, 3rd ed., p. 129. University Press, Cambridge, England, 1952.

Substituting Eqs. (14), (15), (16), (17), and (19) in Eq. (22) yields the following three Equations:

$$\frac{1}{m}\left[(f_0 - B - J)\sin(\Theta + \alpha + \eta) - D\sin\Theta + L\cos\Theta - N\cos(\Theta + \alpha) - mg_1\right]$$

$$= v\cos\Theta\,\dot{\Theta} + \dot{v}\sin\Theta - \frac{v^2}{r}\cos^2\Theta$$

$$+ 2\Omega v\cos\Theta\cos\phi\sin\sigma - \Omega^2 r\cos^2\phi$$

$$\frac{1}{m}\left[(f_0 - B - J)\cos(\Theta + \alpha + \eta)\cos\sigma - D\cos\Theta\cos\sigma - L\sin\Theta\cos\sigma\right.$$

$$\left. + N\sin(\Theta + \alpha)\cos\sigma - mg_2\right]$$

$$= -v\cos\Theta\sin\sigma\,\dot{\sigma} - v\cos\sigma\sin\Theta\,\dot{\Theta} + \dot{v}\cos\Theta\cos\sigma$$

$$+ \frac{v^2}{r}\sin\Theta\cos\Theta\cos\sigma + \frac{v^2}{r}\cos^2\Theta\sin^2\sigma\tan\phi$$

$$- 2\Omega v\cos\Theta\sin\phi\sin\sigma + \Omega^2 r\sin\phi\cos\phi$$

$$\frac{1}{m}\left[(f_0 - B - J)\cos(\Theta + \alpha + \eta)\sin\sigma - D\cos\Theta\sin\sigma - L\sin\Theta\sin\sigma\right.$$

$$\left. + N\sin(\Theta + \alpha)\sin\sigma\right]$$

$$= v\cos\Theta\cos\sigma\,\dot{\sigma} - v\sin\sigma\sin\Theta\,\dot{\Theta} + \dot{v}\cos\Theta\sin\sigma$$

$$+ \frac{v^2}{r}\sin\Theta\cos\Theta\sin\sigma - \frac{v^2}{r}\cos^2\Theta\cos\sigma\sin\sigma\tan\phi$$

$$+ 2\Omega v\cos\Theta\sin\phi\cos\sigma - 2\Omega v\sin\Theta\cos\phi$$

Solving the preceding hree Equations for $\dot{v}$, $\dot{\Theta}$, $\dot{\sigma}$ gives

$$\dot{v} = \frac{1}{m}\left[(f_0 - B - J)\cos(\alpha+\eta) - D\right] + \frac{N}{m}\sin\alpha - g_1(r,\phi)\sin\Theta - g_2(r,\phi)\cos\sigma\cos\Theta$$

$$- \Omega^2 r\left(\frac{1}{2}\sin 2\phi\cos\Theta\cos\sigma - \sin\Theta\cos^2\phi\right) \tag{23}$$

$$\dot{\Theta} = \frac{1}{mv}\left[(f_0 - B - J)\sin(\alpha+\eta) + L - N\cos\alpha\right] - \frac{g_1(r,\phi)}{v}\left(1 - \frac{v^2}{g_1 r}\right)\cos\Theta$$

$$+ \frac{g_2(r,\phi)}{v}\cos\sigma\sin\Theta - 2\Omega\cos\phi\sin\sigma + \frac{\Omega^2 r}{v}\left(\frac{1}{2}\sin 2\phi\sin\Theta\cos\sigma\right.$$

$$\left. + \cos^2\phi\cos\Theta\right) \tag{24}$$

$$\dot{\sigma} = \frac{v}{r}\cos\Theta\sin\sigma\tan\phi - 2\Omega(\sin\phi - \tan\Theta\cos\phi\cos\sigma)$$

$$+ \frac{1}{v}\left[g_2(r,\phi) + \frac{1}{2}\Omega^2 r\sin 2\phi\right]\sec\Theta\sin\sigma \tag{25}$$

Let $\vec{M}$ be the sum of the torques tending to rotate the missile about a transverse axis passing through its C. M., normal to $\vec{v}$ and $\vec{r}$, and let I be the moment of inertia about this axis. Then, from Eq. (5),

$$\frac{d}{dt}[I(\dot{\Theta} + \dot{\alpha})] - \dot{m}\, s_0^2(\dot{\Theta} + \dot{\alpha}) = M \tag{26}$$

Further, let $\vec{T}$ be the sum of the torques (due to a ring of jets placed around the circumference of the missile) tending to rotate the missile about its longitudinal axis, and let I' be the moment of inertia and ω the angular velocity about this axis. Then, from Eq. (8),

$$\frac{d}{dt}(I'\omega) - \dot{m}a(\omega a - v_e) = T \tag{27}$$

where a is the perpendicular distance from the longitudinal axis to a jet and v_e is the exhaust velocity of the jets.

Equations (13), (23), (24), (25), (26), and (27), plus an appropriate control (guidance) equation in pitch, completely describe the motion of the missile relative to the earth.

If (r_0, ϕ_0, θ_0) is the launching point, and (r_I, ϕ_I, θ_I) is the impact point of the missile, then the impact range X_I is given by

$$\frac{X_I}{r_0} = \cos^{-1}[\cos\phi_0 \cos\phi_I \cos(\theta_0 - \theta_I) + \sin\phi_0 \sin\phi_I] \tag{28}$$

Here, it is assumed that $r_I = r_0$.

B. Satellite

The missile, or a part of it, may permanently (ideally) move about the earth in a nearly[b] circular orbit, provided that v and Θ assume appropriate values at the instant of final thrust termination. These values are determined as follows:

If the orbit were perfectly circular, then $\dot{r}(t) \equiv 0$, for $t \geq t'$, where t' = time of final thrust termination. It then follows from Eq. (18a) that

$$\Theta(t) \equiv 0 \qquad (t \geq t') \tag{a}$$

and, consequently,

$$\dot{\Theta}(t) \equiv 0 \qquad (t \geq t') \tag{b}$$

Applying conditions a and b to Eq. (24) yields

$$v = \Omega r \cos\phi \sin\sigma + (g_1 r - \Omega^2 r^2 \cos^2\phi \cos^2\sigma)^{1/2} \qquad (t \geq t')$$

[b] The oblateness of the earth prevents the orbit from being perfectly circular and also prevents it from lying in a plane.

which is the speed of the satellite relative to the earth. Thus, if the satellite is to describe a nearly circular orbit at a distance $\sim r$ from the earth's center, then

$$v(t') = \Omega r(t') \cos\phi(t') \sin\sigma(t') + [g_1(t')r(t') - \Omega^2 r^2(t') \cos^2\phi(t') \cos^2\sigma(t')]^{1/2} \tag{29a}$$

$$\Theta(t') = 0 \tag{29b}$$

Equations (13), (23), (24), and (25), with $f_0 = B = J = L = N = 0$, therefore describe the motion of the satellite relative to the earth, subject to the "initial" conditions

$$\left.\begin{array}{c} r(t') \\ \phi(t') \\ \theta(t') \\ \text{Eq. (29a)} \\ \text{Eq. (29b)} \\ \sigma(t') \end{array}\right\} \tag{30}$$

If $r(t')$ is large (e.g., $r(t') \simeq 1.25\,R$), then

$$g_1 \simeq \frac{GM}{r^2}$$

$$g_2 \simeq 0$$

$$D \simeq 0$$

In such a case, the orbit could be considered circular, so that the equations of motion would be

$$\left.\begin{aligned} \dot{\phi} &= \frac{v}{r(t')} \cos \sigma \\ \dot{\theta} &= \frac{v}{r(t')} \frac{\sin \sigma}{\cos \phi} \\ \dot{\sigma} &= \frac{v}{r(t')} \sin \sigma \tan \phi - 2\Omega \sin \phi + \frac{1}{2} \frac{\Omega^2 r(t')}{v} \sin 2\phi \sin \sigma \end{aligned}\right\} \qquad (31)$$

where

$$v = \Omega r(t') \cos \phi \sin \sigma + \left[\frac{GM}{r(t')} - \Omega^2 r^2(t') \cos^2 \phi \cos^2 \sigma\right]^{1/2} \qquad (32)$$

The values of $\phi(t')$ and $\sigma(t')$ determine the *initial* angle that the orbit makes with the equatorial plane. Thus, in Fig. 8, the orbital angular-velocity vector $\vec{\omega}$ of the satellite makes an angle ψ with the rotational axis of the earth; λ is the angle between the positive x axis and the projection of $\vec{\omega}$ in the x-z plane.

Then

$$\vec{\omega} = \vec{i}\omega \sin \psi \cos \lambda + \vec{j}\omega \cos \psi + \vec{k}\omega \sin \psi \sin \lambda$$

Initially, $\vec{r}$ lies in the plane of the orbit, so that

$$\omega = |\vec{\omega}| = \frac{v}{r} = \left(\frac{g_1}{r}\right)^{1/2}$$

In accordance with Eq. (11),

$$\vec{\omega} = \vec{r}_0 \, \omega_r + \vec{\phi}_0 \, \omega_\phi + \vec{\theta}_0 \omega_\theta$$

where

$$\omega_r = \omega \sin\psi \cos\phi \cos(\lambda - \theta) + \omega \cos\psi \sin\phi \tag{33a}$$

$$\omega_\phi = -\omega \sin\psi \sin\phi \cos(\lambda - \theta) + \omega \cos\psi \cos\phi \tag{33b}$$

$$\omega_\theta = \omega \sin\psi \sin(\lambda - \theta) \tag{33c}$$

Now,

$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

where d/dt refers to inertial space. If the earth is taken as the reference, the equation above becomes

$$\frac{d\vec{r}}{dt} + \vec{\Omega} \times \vec{r} = \vec{\omega} \times \vec{r} \tag{34}$$

But, from Eq. (12),

$$\frac{d\vec{r}}{dt} = \vec{r}_0 \dot{r} + \vec{\phi}_0 v \cos\Theta \cos\sigma + \vec{\theta}_0 v \cos\Theta \sin\sigma$$

Further, $\vec{\Omega} \times \vec{r} = -\vec{\theta}_0 \Omega r \cos\phi$, and $\vec{\omega} \times \vec{r} = \vec{\phi}_0 r\omega_\theta - \vec{\theta}_0 r\omega_\phi$. Substituting in Eq. (34) and equating components yields

$$\dot{r} = 0$$

$$\omega_\theta = \frac{v}{r} \cos\Theta \cos\sigma$$

$$\omega_\phi = \Omega \cos\phi - \frac{v}{r} \cos\Theta \sin\sigma$$

However, for $t \simeq t'$, $\Theta = 0$. Combining the last two Equations with Eqs. (33b) and (33c) gives

$$\frac{v}{r} \cos \sigma = \omega \sin \psi \sin (\lambda - \theta) \tag{35a}$$

$$\Omega \cos \phi - \frac{v}{r} \sin \sigma = \omega \cos \psi \cos \phi - \omega \sin \psi \sin \phi \cos (\lambda - \theta) \tag{35b}$$

Since $\vec{\omega} \cdot \vec{r} = 0$ initially, $\omega_r = 0$. Then, from Eq. (33a),

$$\cos \psi \sin \phi = - \sin \psi \cos \phi \cos (\lambda - \theta) \tag{35c}$$

Combining Eqs. (35b) and (35c) yields the desired relation

$$\cos \psi = \left(\frac{r}{g_1}\right)^{1/2} \left(- \frac{v}{r} \sin \sigma \cos \phi + \Omega \cos^2 \phi\right)$$

Consequently,

$$\begin{aligned} \cos \psi(t') &= H(t') & (-\pi < \sigma(t') < 0) \\ &= - H(t') & (0 < \sigma(t') < \pi) \end{aligned}$$

where

$$H(t') = \left[\frac{r(t')}{g_1(t')}\right]^{1/2} \left[\frac{-v(t')}{r(t')} \sin \sigma(t') \cos \phi(t') + \Omega \cos^2 \phi(t')\right]$$

Here, $v(t')$ is given by Eq. (29a).

VI. APPROXIMATE LOCATION OF THE MISSILE IMPACT POINT FOR A ROTATING EARTH

The exact location of the missile impact point is obtained by integrating Eqs. (13), (23) to (27), and the appropriate control equation from takeoff ($t = t_0$) to impact ($t = t_I$). It is possible to describe the approximate location of this point by integrating the equations only up to $t = t'$, the time at which the guidance is terminated (the thrust having terminated at $t \leq t'$). The impact point can then be determined from the values of the missile position and velocity at $t = t'$: $r(t')$, $\phi(t')$, $\theta(t')$, $v(t')$, $\Theta(t')$, and $\sigma(t')$.

It is assumed that the earth is a perfect sphere (inverse-square force law), and that the missile moves through a vacuum during the interval $t' \leq t \leq t_I$. Hence, it will describe an ellipse (in inertial space) during this time (see Fig. 9). Here, f_1 and f_2 are the foci of the ellipse, which lies in a plane fixed in inertial space. Let Φ be the angular displacement of the missile from the line passing through f_1 and f_2. Then the equation of the ellipse is

$$\frac{1}{r} = \frac{1}{s}(1 - e \cos \Phi) \tag{36}$$

where

$$s = \frac{v_i(t')\, r^2(t') \cos^2 \Theta_i(t')}{g_0 r_0^2}$$

$$e^2 = 1 - \frac{2s}{r(t')} + \frac{s^2}{r^2(t') \cos^2 \Theta_i(t')}$$

The subscript i denotes that the associated quantity is measured in inertial space. Here, e is the eccentricity of the ellipse, and g_0 and r_0 are the values of g and r measured at the earth's surface. (The impact altitude is assumed equal to the launching altitude.)

The impact point $\phi_i(t_I)$, $\theta_i(t_I)$ is a function of $r(t')$, $\phi_i(t')$, $\theta_i(t')$, $v_i(t')$, $\Theta_i(t')$, $\sigma_i(t')$ (refer to the spherical right triangle in Fig. 10). From Fig. 10,

$$\cos \sigma_i(t') = \frac{\tan \eta}{\tan \chi} \tag{37a}$$

$$\sin \sigma_i(t') = \frac{\sin \zeta}{\sin \chi} \tag{37b}$$

Substituting this in Eq. (37b) yields

$$\cos \zeta = \cos \bar{\phi}_i(t') \cos \phi_i(t_I) \cos [\theta_i(t') - \theta_i(t_I)] + \sin \bar{\phi}_i(t') \sin \phi_i(t_I)$$

From Eq. (28),

$$\cos \bar{\phi}_i(t') \cos \phi_i(t_I) \cos [\theta_i(t') - \theta_i(t_I)] + \sin \bar{\phi}_i(t') \sin \phi_i(t_I) = [1 - \sin^2\sigma_i(t') \sin^2\chi]^{1/2} \tag{38}$$

From Eq. (28),

$$\cos \phi_i(t') \cos \phi_i(t_I) \cos [\theta_i(t') - \theta_i(t_I)] + \sin \phi_i(t') \sin \phi_i(t_I) = \cos \chi \tag{39}$$

Combining Eqs. (38) and (39) gives

$$\sin \phi_i(t_I) = \frac{[1 - \sin^2\sigma_i(t') \sin^2\chi]^{1/2} \cos \phi_i(t') - \cos \bar{\phi}_i(t') \cos \chi}{\sin [\bar{\phi}_i(t') - \phi_i(t')]} \tag{40}$$

But $\eta = \bar{\phi}_i(t') - \phi_i(t')$. Hence, from Eq. (37a),

$$\bar{\phi}_i(t') = \phi_i(t') + \tan^{-1} [\cos \sigma_i(t') \tan \chi]$$

Substituting this in Eq. (40) yields

$$\sin \phi_i(t_I) = \cos \phi_i(t') \cos \sigma_i(t') \sin \chi + \sin \phi_i(t') \cos \chi \tag{41}$$

The value of $\theta_i(t_I)$ can then be obtained from Eq. (39):

$$\theta_i(t_I) = \theta_i(t') + \cos^{-1}[\sec\phi_i(t')\sec\phi_i(t_I)\cos\chi - \tan\phi_i(t')\tan\phi_i(t_I)] \qquad (0 < \sigma_i(t') < \pi)$$

$$= \theta_i(t') - \cos^{-1}[\ ] \qquad (-\pi < \sigma_i(t') < 0)$$

$$= \theta_i(t') \qquad (\sigma_i(t') = 0, \pi, -\pi) \quad (42)$$

The angle χ is obtained as follows: from Eq. (36),

$$\Phi(t') = -\cos^{-1}\frac{1}{e}\left[1 - \frac{s}{r(t')}\right]$$

$$\Phi(t_I) = \cos^{-1}\frac{1}{e}\left(1 - \frac{s}{r_0}\right)$$

But $\chi = |\Phi(t')| + \Phi(t_I)$. Hence,

$$\chi = \cos^{-1}\frac{1}{e}\left[1 - \frac{s}{r(t')}\right] + \cos^{-1}\frac{1}{e}\left(1 - \frac{s}{r_0}\right) \quad (43)$$

To obtain the desired impact coordinates $\phi(t_I)$, $\theta(t_I)$ (measured relative to the earth), a relation must be obtained between the position and velocity coordinates of the missile, as measured in inertial space, and earth-fixed coordinates. In Fig. 11, the subscript i denotes inertial space, and $\Omega(t - t')$ is the angle through which the earth has rotated in the time $t - t'$. Here, $t \geq t'$. Now,

$$\left(\frac{d}{dt}\right)_i \vec{r} = \frac{d\vec{r}}{dt} + \vec{\Omega} \times \vec{r}$$

But $\vec{v} = d\vec{r}/dt$. Therefore,

$$\vec{v}_i = \vec{v} + \vec{\Omega} \times \vec{r} \tag{44}$$

Now, $\vec{\Omega} = \vec{r}_0\, \Omega \sin\phi + \vec{\phi}_0\, \Omega \cos\phi$. Substituting this expression and Eq. (10) into Eq. (44) yields the three Equations

$$v_i \sin\Theta_i = v \sin\Theta$$

$$v_i \cos\Theta_i \cos\sigma_i = v \cos\Theta \cos\sigma$$

$$v_i \cos\Theta_i \sin\sigma_i = v \cos\Theta \sin\sigma - r\,\Omega \cos\phi$$

which combine to give

$$\left.\begin{aligned} v_i^2 &= v^2 + \Omega r \cos\phi\,(\Omega r \cos\phi - 2v \cos\Theta \sin\sigma) \\ \sin\Theta_i &= \frac{v}{v_i} \sin\Theta \\ \tan\sigma_i &= \tan\sigma - \Omega\,\frac{r}{v} \sec\Theta \sec\sigma \cos\phi \end{aligned}\right\} \tag{45}$$

From Fig. 11,

$$x_i = x \cos\Omega\,(t - t') + z \sin\Omega\,(t - t')$$

$$y_i = y$$

$$z_i = -x \sin\Omega\,(t - t') + z \cos\Omega\,(t - t')$$

Substituting Eqs. (9) into the expressions above yields the three Equations

$$\cos\theta_i \cos\phi_i = \cos\theta \cos\phi \cos\Omega\,(t - t') + \sin\theta \cos\phi \sin\Omega\,(t - t')$$

$$\sin \phi_i = \sin \phi$$

$$\sin \theta_i \cos \phi_i = -\cos \theta \cos \phi \sin \Omega (t - t') + \sin \theta \cos \phi \cos \Omega (t - t')$$

which combine to give

$$\left.\begin{aligned} \phi_i &= \phi \\ \theta_i &= \theta - \Omega (t - t') \end{aligned}\right\} \quad (46)$$

Equations (45) and (46) are the desired relations. The quantity $t - t'$ is obtained as follows: from Eq. (36),

$$-\frac{\dot{r}}{r^2} = \frac{e}{s} \sin \Phi \cdot \dot{\Phi}$$

Combining this and Eq. (36) with the Equations

$$v_i \sin \Theta_i = \dot{r}$$

$$v_i \cos \Theta_i = r \dot{\Phi}$$

$$P = r v_i \cos \Theta_i = \text{angular momentum} = \text{constant}$$

yields

$$\dot{r} = \frac{dr}{dt} = + \frac{P}{r}\left[(e^2 - 1)\frac{r^2}{s^2} + \frac{2r}{s} - 1\right]^{1/2} \qquad (t' \leq t < t_s) \qquad (47a)$$

$$= -\frac{P}{r}\left[(e^2 - 1)\frac{r^2}{s^2} + \frac{2r}{s} - 1\right]^{1/2} \qquad (t_s < t \leq t_l) \qquad (47b)$$

Here, t_s = time at which the missile reaches the summit (see Fig. 9). Integrating Eq. (47a) from t' to t_s and Eq. (47b) from t_s to t $(t_s < t \leq t_I)$ and combining the two results gives

$$t - t' = \frac{s}{P\left(\frac{1}{e} - e\right)} \left\{ r(t)\left[1 - \frac{1}{e^2}\left(1 - \frac{s}{r(t)}\right)^2\right]^{1/2} + r(t')\left[1 - \frac{1}{e^2}\left(1 - \frac{s}{r(t')}\right)^2\right]^{1/2} \right.$$

$$+ \frac{s}{e(1-e^2)^{1/2}} \sin^{-1}\left[\frac{1}{s}\left(e - \frac{1}{e}\right) r(t) + \frac{1}{e}\right]$$

$$+ \frac{s}{e(1-e^2)^{1/2}} \sin^{-1}\left[\frac{1}{s}\left(e - \frac{1}{e}\right) r(t') + \frac{1}{e}\right]$$

$$\left. + \frac{\pi s}{e(1-e^2)^{1/2}} \right\} \tag{48}$$

Here, $P = r(t')\, v_I(t') \cos \Theta_I(t')$.

Equations (43), (45), (46), and (48), used in conjunction with Eqs. (41) and (42), yield the desired impact point $\phi(t_I)$, $\theta(t_I)$.

If, for simplicity, it is assumed that the earth does not rotate during the interval $t_0 \leq t \leq t'$, then the missile path, during this interval, will lie in a plane relative to the earth; therefore, the equations to be integrated will simply be: Eq. (18a); the expression $\dot{x} = (r_0/r) v \cos \Theta$ (where x = ground range measured along the surface of the earth); Eqs. (23) and (24) (with $\Omega = 0$); Eq. (26); and a control equation in pitch. However, only the quantities $r(t')$, $v(t')$ and $\Theta(t')$ are obtainable from the solutions. The remaining quantities $\phi(t')$, $\theta(t')$, and $\sigma(t')$ can be determined from the known values of $\phi(t_0)$, $\theta(t_0)$, $\sigma(t_0)$, and $x(t')$. Thus, analogous to Eqs. (41) and (42),

$$\phi(t') = \sin^{-1}\left[\cos \phi(t_0) \cos \sigma(t_0) \sin \frac{x(t')}{r_0} + \sin \phi(t_0) \cos \frac{x(t')}{r_0}\right] \tag{49}$$

$$\theta(t') = \theta(t_0) + \cos^{-1}\left[\sec\phi(t_0)\sec\phi(t')\cos\frac{x(t')}{r_0} - \tan\phi(t_0)\tan\phi(t')\right] \qquad (0 < \sigma(t_0) < \pi)$$

$$= \theta(t_0) - \cos^{-1}[\ \ ] \qquad (-\pi < \sigma(t_0) < 0)$$

$$= \theta(t_0) \qquad (\sigma(t_0) = 0, \pi, -\pi) \qquad (50)$$

Consider the spherical right triangles in Fig. 12. From this Figure,

$$\tan\sigma(t_0) = \frac{\tan\beta_1}{\sin\phi(t_0)}$$

$$\tan\sigma(t') = \frac{\tan\beta_2}{\sin\phi(t')}$$

$$\frac{\tan\phi(t_0)}{\sin\beta_1} = \frac{\tan\phi(t')}{\sin\beta_2}$$

These combine to give

$$\sigma(t') = \sin^{-1}[\cos\phi(t_0)\sin\sigma(t_0)\sec\phi(t')] \qquad (51)$$

The earlier t' occurs, the more nearly will the impact point $\phi(t_I)$, $\theta(t_I)$, as obtained from these values of $r(t')$, $\phi(t')$, $\theta(t')$, $v(t')$, $\Theta(t')$, and $\sigma(t')$, coincide with the $\phi(t_I)$, $\theta(t_I)$ determined in the preceding paragraph.

Table 1. Nomenclature

a = perpendicular distance from the missile longitudinal axis to a jet located on the circumference.

$\vec{B}$ = base drag force.

C. M. = center of mass of the missile.

$\vec{c}$ = velocity of m_1 with respect to the C. M. of the missile.

$\vec{D}$ = drag force.

e = eccentricity of ellipse (Fig. 9).

$\vec{F}$ = external forces acting on m_1 and m.

$\vec{f_i}$ = thrust from the ith jet.

$\vec{f_0}$ = vacuum thrust.

f_1, f_2 = foci of ellipse (Fig. 9).

G = universal gravitation constant.

$\vec{g}$ = acceleration of gravity.

g_0 = value of gravity acceleration at the launching point.

I = moment of inertia of the missile.

$\vec{i}, \vec{j}, \vec{k}$ = unit vectors pointing in the direction of increasing x, y, z, respectively.

$\vec{J}$ = jet vane and/or exhaust-obstruction effect.

$\left.\begin{matrix} K_1 \\ K_2 \end{matrix}\right\}$ = constants which are a measure of the earth's oblateness.

$\vec{L}$ = angular-momentum vector (Sec. III).

$\vec{L}$ = lift force (Sec. V).

M = mass of the earth.

$\vec{M}$ = torques taken about a transverse axis through the C. M. of the missile.

m = mass of the missile.

Table 1 (Cont'd)

m_1 = mass of a quantity of exhaust expelled from the missile.

N = total number of jets.

$\vec{N}$ = lift force of the rudder.

P = position.

p = orbital angular momentum = constant.

$\vec{p}$ = total linear momentum of m_1 and m, measured in $\Sigma\ (x,\ y,\ z)$.

R = equatorial radius of the earth.

$r = |\vec{r}|$.

$\vec{r}$ = position vector drawn from the center of the earth to the C. M. of the missile.

$\left.\begin{matrix} r_I \\ \phi_I \\ \theta_I \end{matrix}\right\}$ = impact point of the missile.

$\left.\begin{matrix} r_0 \\ \phi_0 \\ \theta_0 \end{matrix}\right\}$ = launching point of the missile.

$\left.\begin{matrix} \vec{r}_{0_i} \\ \vec{\theta}_{0_i} \\ \vec{\omega}_0 \end{matrix}\right\}$ = unit vectors.

$\vec{r}_0, \vec{\phi}_0, \vec{\theta}_0$ = unit vectors pointing in th ection of increasing r, ϕ, θ, respectively.

s = a constant determined by the missile position and velocity at $t = t'$.

s_0 = average distance from the C. M. of the missile to the base of the missile.

$\left.\begin{matrix} \vec{s} \\ \vec{\rho} \\ \vec{r}_1 \\ \vec{R} \end{matrix}\right\}$ = position vectors.

Table I (Cont'd)

$\left.\begin{array}{l}\vec{s}_1\\ \vec{\theta}_1\\ \vec{\phi}_1\end{array}\right\}$ = unit vectors

t = time.

t' = time of final thrust termination and/or time of pitch-guidance termination.

t_I = impact time of the missile.

t_s = time at which the missile reaches the summit of its trajectory.

t_0 = instant of time immediately before m_1 is expelled (Sec. II).

$t_0 + h$ = instant of time immediately after m_1 is expelled (Sec. II).

t_0 = launching time of the missile (Sec. VI).

V = gravity potential function for an oblate-spheroid earth.

v = $|\vec{v}|$.

$\vec{v}(t)$ = velocity of the missile C. M. = $d\vec{r}/dt$.

v_e = exhaust velocity of jets situated around the circumference of the missile.

x = ground range measured along the surface of the earth.

x_I = impact ground range of the missile.

x_i, y_i, z_i = Cartesian coordinates measured in inertial space.

α = angle of attack.

$\left.\begin{array}{l}\beta_1\\ \beta_2\end{array}\right\}$ = arcs of great circles on the earth's surface (Fig. 12).

$\vec{\gamma}$ = angular-velocity vector.

η = angle between the motor thrust and the missile axis (Sec. V).

Table 1 (Cont'd)

$\left.\begin{array}{l}\eta\\ \chi\\ \zeta\\ \bar{\phi}_i\end{array}\right\}$ = arcs of great circles on the earth's surface (Sec. VI).

Θ = inclination of $\vec{v}$ to the local horizontal.

θ = longitude.

λ = angle between the $+x$ axis and the projection of $\vec{\omega}$ in the xz plane (Fig. 8).

$\Sigma\,(x, y, z)$ = set of orthogonal axes fixed in inertial space.

$\Sigma'(x', y', z')$ = set of orthogonal axes fixed in the earth.

σ = angle between the local horizontal and the local meridian.

$\vec{T}$ = torques taken about the longitudinal axis of the missile.

ϕ = latitude.

$\left.\begin{array}{l}\phi_i\\ \theta_i\\ v_i\\ \Theta_i\\ \sigma_i\end{array}\right\}$ = quantities measured in inertial space.

Φ = angle between r and the line through f_1 and f_2.

ψ = angle between $\vec{\omega}$ and $\vec{\Omega}$ (Fig. 8).

$\vec{\Omega}$ = spin angular-velocity vector of the earth.

$\vec{\omega}$ = angular-velocity vector of the missile about the longitudinal axis (Sec. III).

$\vec{\omega}$ = orbital angular velocity of the satellite (Sec. V-B and Fig. 8).

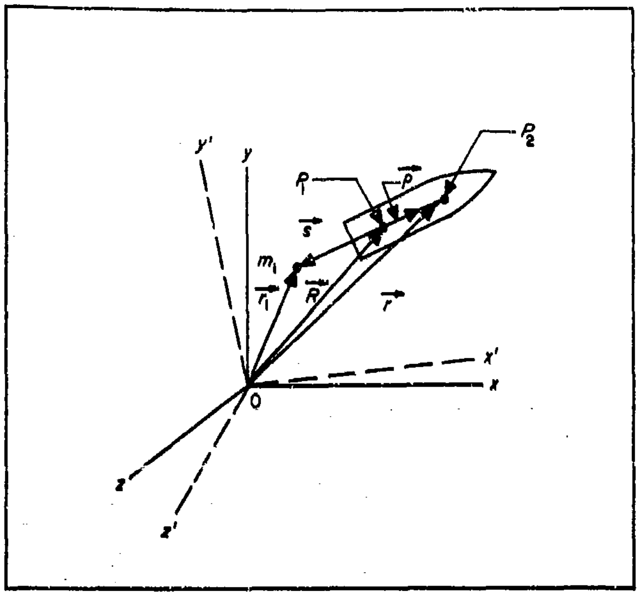

Fig. 1. Definition of the Earth-Fixed and Inertial Coordinate Systems

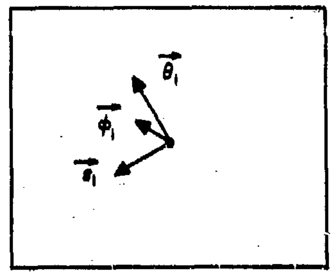

Fig. 2. Definition of the Unit Vectors $\vec{\theta}_1$, $\vec{\phi}_1$, and $\vec{s}_1$

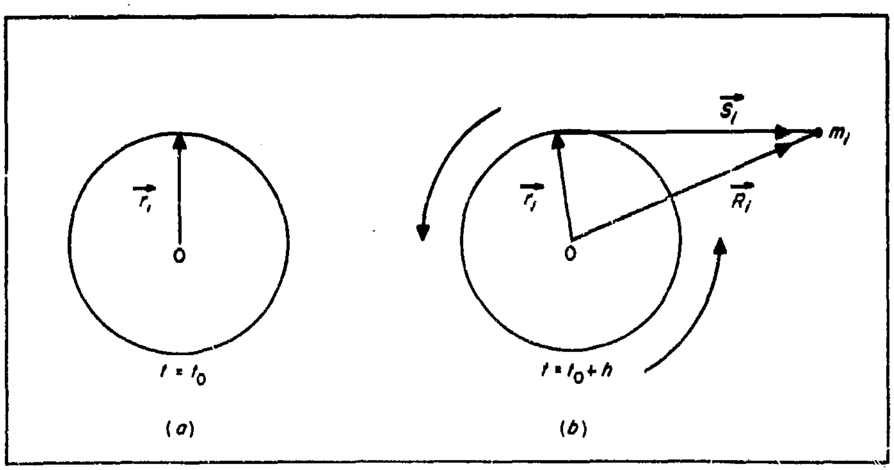

Fig. 3. End View of Missile

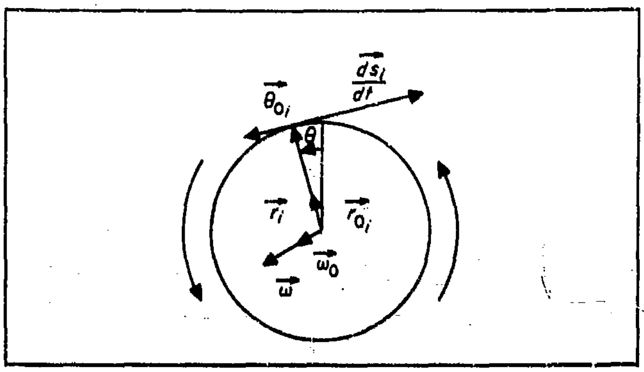

Fig. 4. Rotation of Missile About Its Longitudinal Axis ($\omega = \dot{\theta}$)

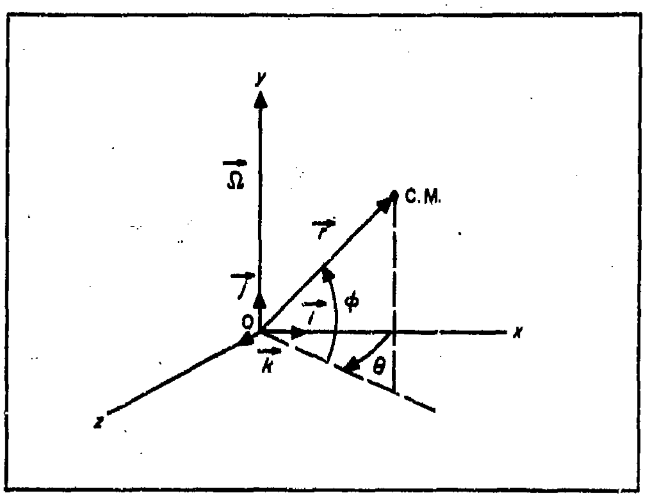

Fig. 5. Definition of Position Coordinates

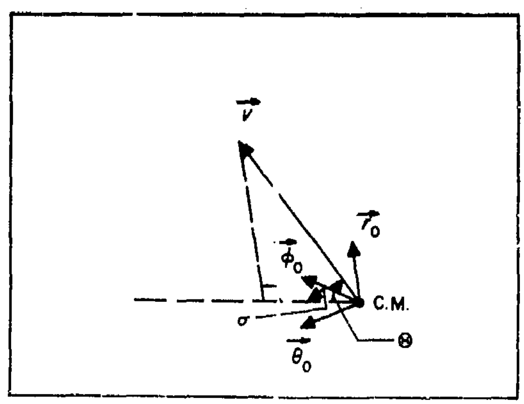

Fig. 6. Definition of Velocity Coordinates

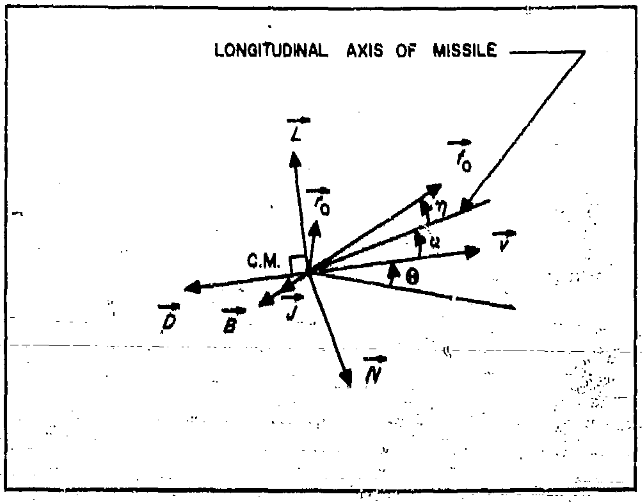

Fig. 7. Nongravitational Forces Acting on Missile

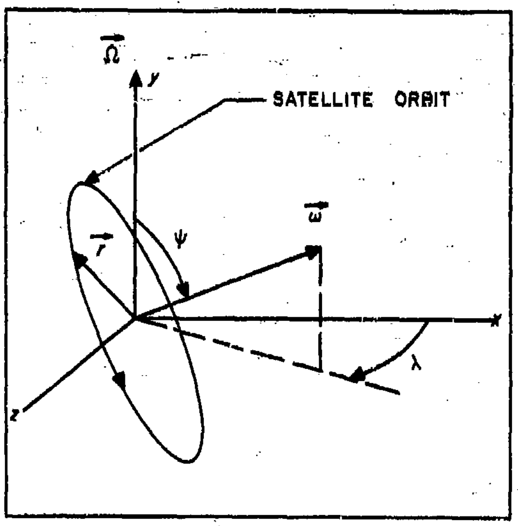

Fig. 8. Inclination of Satellite Orbit to Equatorial Plane

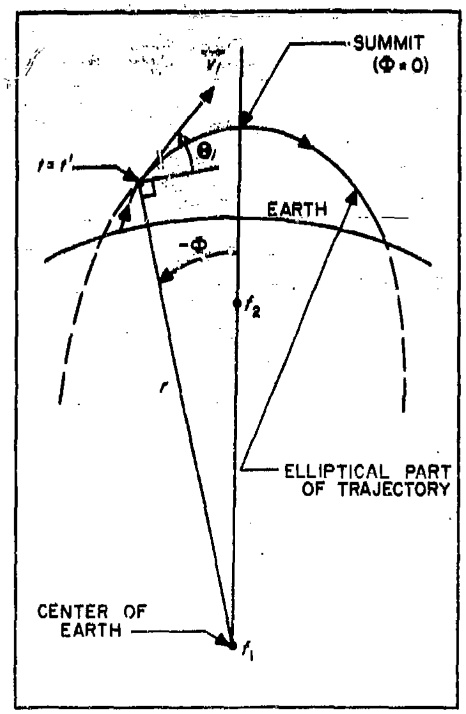

Fig. 9. Elliptical Path Described by Missile

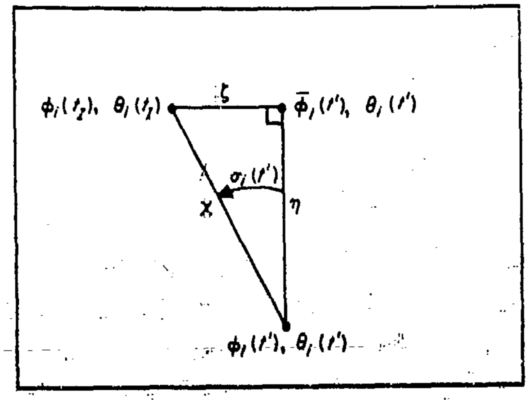

Fig. 10. Missile Impact Point and Projection of Missile Shutoff Point on Earth's Surface

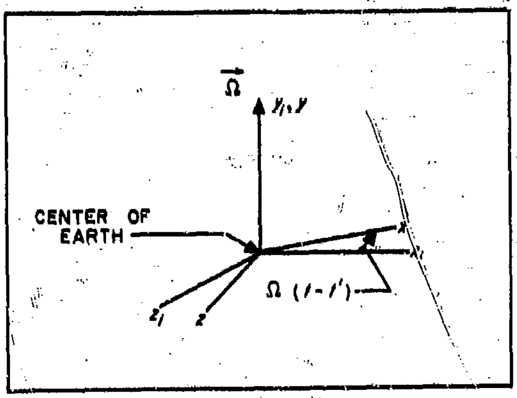

Fig. 11. Rotation of the Earth-Fixed System Relative to Inertial Space

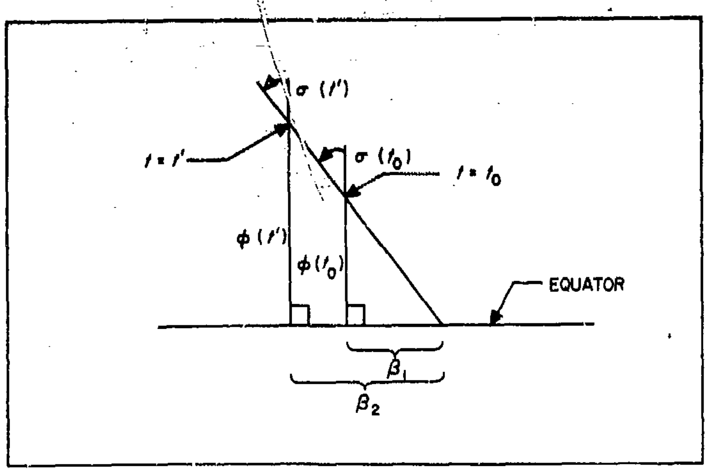

Fig. 12. Spherical Triangles on Earth's Surface Relating $\sigma(t')$ and $\sigma(t_0)$